

Noise sources & Information Theory

Thermal Noise / Resistor Noise:-

This noise is due to the random motion of free electrons in a conducting medium such as a resistor. The conduction electrons in a conductor can wander randomly through the entire volume of the conductor. As a result of scattered thermal energy, the randomness in the motion of free electrons is much more increased. This results in non-uniform charge distribution throughout the conductor, i.e. at any instant of time, excess of electrons may appear at one point relative to the other along the length of the conductor.

$$P_n = kTB \text{ Watts}$$

Where k = Boltzmann constant

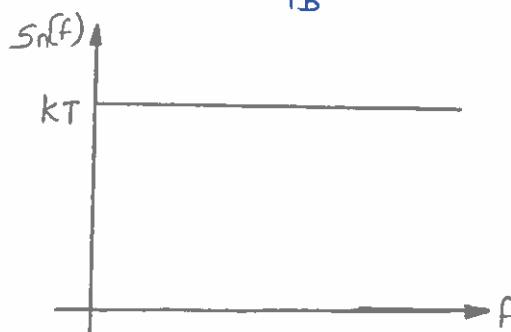
$$= 1.38 \times 10^{-23} \text{ J/}^\circ\text{K.}$$

B = Bandwidth of noise spectrum (Hz)

T = Noise Temperature in $^\circ\text{K}$.

Since, the power spectral density is power per unit Bandwidth, thermal noise PSD is

$$S_n = P_n/B = kT \text{ Watts/Hz}$$



The two sided thermal noise PSD is $\frac{P_n}{2B} = \frac{kT}{2}$ W/Hz. It is found that the thermal noise PSD is independent of frequency, similar to white noise.

Thus, thermal noise is also considered as white noise over a wide range of frequency.

$$\frac{V^2}{4R} \text{ or } \frac{I^2 R}{4} \text{ watts}$$

Thus a noisy resistor delivers a maximum of $\frac{V_n^2}{4R}$ or $\frac{I_n^2 R}{4}$, where V_n and I_n are noise voltage and current respectively, since, thermal noise power is kTB watts.

$$\frac{V_n^2}{4R} = kTB$$

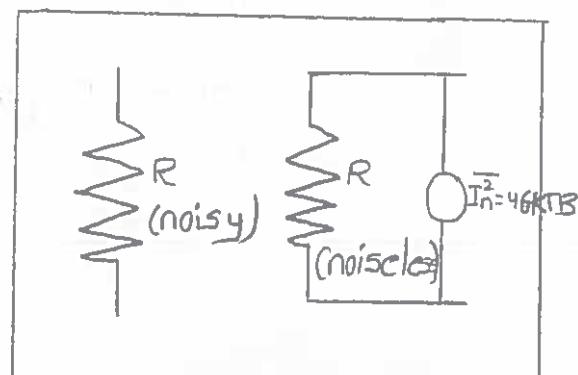
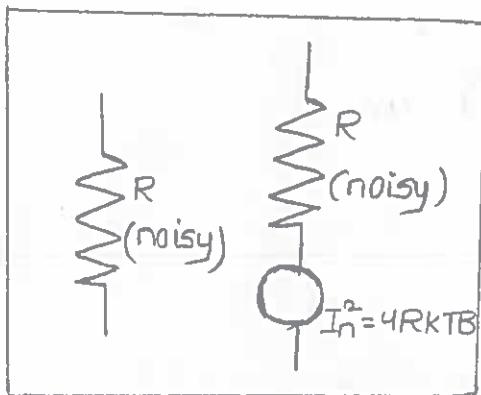
$$\Rightarrow V_n^2 = 4RkTB$$

Thus a noisy resistor R can be represented as noiseless resistor R in series with a noise voltage source and current with mean square value.

$$\overline{V_n^2} = 4RkTB \text{ Volts}^2$$

$$\text{Hence } \frac{I_n^2 R}{4} = kTB$$

$$\Rightarrow I_n^2 = \frac{4kTB}{R} = 4GkTB \cdot \text{amp}^2$$



Problem:

TWO resistors $30\text{k}\Omega$ and $50\text{k}\Omega$ are at room temperature of 30°C . For a bandwidth of 1MHz determine the thermal noise voltage of

- each resistor
- two resistors in series and
- two resistors in parallel.

Sol:

Here

$$T_0 = 30^\circ\text{C} = 273 + 30 \quad T = 303^\circ\text{K}$$

$$kT = 1.38 \times 10^{-21} \times 303 = 4.181 \times 10^{-21} \text{ W/Hz at } 303^\circ\text{K.}$$

(i) For the $30\text{k}\Omega$ resistor, thermal noise voltage.

$$E_N = \sqrt{4kTR_{30}B} = \sqrt{4 \times 4.181 \times 10^{-21} \times 30 \times 10^3 \times 10^6} \\ = 2.24 \times 10^{-5} = 0.224 \mu\text{V} = 22.4 \mu\text{V}$$

For $50\text{k}\Omega$ resistor, the thermal noise voltage is

$$E_N = \sqrt{4kTR_{50}B} = \sqrt{4 \times 4.181 \times 10^{-21} \times 50 \times 10^3 \times 10^6} = 28.9 \mu\text{V}$$

$$R_s = 30 + 50 = 80\text{k}\Omega$$

$$E_N = \sqrt{4kTR_sB} = \sqrt{4 \times 4.181 \times 10^{-21} \times 8 \times 10^3 \times 10^6} \\ = 11.5 \mu\text{V}$$

(ii) In parallel

$$R_p = \frac{30 \times 50}{30 + 50} = \frac{1500}{80} = 18.75\text{k}\Omega$$

$$E_N = \sqrt{4kTR_pB} = \sqrt{4 \times 4.181 \times 10^{-21} \times 18.75 \times 10^3 \times 10^6} \\ = 17.7 \mu\text{V}$$

Entropy

Consider a communication system in which allowable messages are $M_1, M_2 \dots$ with probabilities of occurrence $P_1, P_2 \dots$

let the transmitter select message M_k of probability P_k .

Then by way of definition by information the system has conveyed an amount of information I_k is given by

$$I_k = \log_2 \left(\frac{1}{P_k} \right)$$

If 2 is the base then the unit information is bit. If e is the base then the unit of information is not for the decimal logarithm this unit is called decit, decit is also termed as hartley.

$$H = P_1 \log_2 \frac{1}{P_1} + P_2 \log_2 \frac{1}{P_2}$$

If there are N messages $M_1, M_2, M_3 \dots$ and each message is equally likely then the probability of M_i

$$P(M_i) = \frac{1}{N} P(M_1)$$

$$\begin{aligned} H &= \frac{1}{N} [\log_2(n) + \log_2(n) \dots] = \frac{1}{N} \times n (\log_2(n)) \\ &= \log_2 n \end{aligned}$$

When messages are equally likely otherwise.

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$$H = \pi \times \log_2(n)$$

π = some co-efficient which is < 1

Hence maximum information per message occurs when messages are equal likely.

Conditional Entropy - $H\left[\frac{Y}{X}\right]$

Consider that an information source transmits a symbol x_i . One of the permissible destination probability symbols y_i is received with a given probability. The entropy associated with this transmitted as the condition covers the set of all source (alphabet) is defined entropy $H\left(\frac{Y}{X}\right)$.

As a measure of information about the receiving side where it is known that x is transmitted.

$$H\left(\frac{Y}{X}\right) = \sum_{i=1}^M \sum_{j=1}^N p(x_i; y_j) \frac{1}{\log p(y_j/x_i)}$$

$$\therefore 0 \leq H\left(\frac{Y}{X}\right) \leq H(Y)$$

This represents the average uncertainty about a received signal given that x was transmitted.

Find the PSD of thermal noise voltage across the terminals 1-2 for the following circuit.

sol:

The Thermal noise PSD is $G(f) = 2kT R(f)$. consider the admittance $Y_{12}(f)$.

$$= \frac{1}{2} + \frac{1}{2+j2\omega} + j2\omega - \frac{1+j\omega+1+j2\omega \cdot (2+j2\omega)}{2(1+j\omega)}$$

$$= \frac{2+j\omega+4j\omega-4\omega^2}{2(1+j\omega)} = \frac{(2-4\omega^2)+5j\omega}{2(1+j\omega)}$$

The corresponding impedance $Z_{12}(f) = \frac{1}{Y_{12}(f)}$

$$\therefore Z_{12}(f) = \frac{2(1+j\omega)}{(2-4\omega^2)+5j\omega} = \frac{2(1+j\omega)}{2-4\omega^2+5j\omega} \times \frac{(2-4\omega^2)-5j\omega}{(2-4\omega^2)-5j\omega}$$

The real part of the above $Z_{12}(f)$ is

$$\frac{2(2-4\omega^2)+10\omega^2}{(2-4\omega^2)+25\omega^2} = \frac{2\omega^2+4}{16\omega^4+9\omega^2+4}$$

\therefore The required thermal noise PSD is

$$G(f) = 2kT \cdot R(f) = \frac{2kT(2\omega^2+4)}{16\omega^4+9\omega^2+4}$$

Problem:

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The noise matrix of channel is given as

$$\begin{matrix} & Y_1 & Y_2 & Y_3 \\ X_1 & \begin{bmatrix} 0.4 & 0.4 & 0.2 \end{bmatrix} \\ X_2 & \begin{bmatrix} 0.2 & 0.4 & 0.4 \end{bmatrix} \\ X_3 & \begin{bmatrix} 0.4 & 0.2 & 0.4 \end{bmatrix} \end{matrix}$$

The inputs for the channel are source messages x_1, x_2 and x_3 with respective probabilities of 0.7, 0.2 and 0.1.

Find the entropy of the source X , receiver, joint entropy of the system, conditional entropies and the amount of mutual information.

Sol:-

$$P(Y|X) = \begin{matrix} & Y_1 & Y_2 & Y_3 \\ X_1 & \begin{bmatrix} 0.4 & 0.4 & 0.2 \end{bmatrix} \\ X_2 & \begin{bmatrix} 0.2 & 0.4 & 0.4 \end{bmatrix} \\ X_3 & \begin{bmatrix} 0.4 & 0.2 & 0.4 \end{bmatrix} \end{matrix}$$

$$P(X, Y) = \begin{bmatrix} 0.4 \times 0.7 & 0.4 \times 0.7 & 0.2 \times 0.7 \\ 0.2 \times 0.2 & 0.4 \times 0.2 & 0.4 \times 0.2 \\ 0.4 \times 0.1 & 0.2 \times 0.1 & 0.4 \times 0.1 \end{bmatrix}$$

$$P(X, Y) = \begin{matrix} & Y_1 & Y_2 & Y_3 \\ X_1 & \begin{bmatrix} 0.28 & 0.28 & 0.14 \end{bmatrix} \\ X_2 & \begin{bmatrix} 0.04 & 0.08 & 0.08 \end{bmatrix} \\ X_3 & \begin{bmatrix} 0.04 & 0.02 & 0.04 \end{bmatrix} \end{matrix}$$

Entropy of the source:

$$P(X_1) = 0.7, P(X_2) = 0.2$$

$$P(X_3) = 0.1$$

$$H(X) = -[0.7 \log 0.7 + 0.2 \log 0.2 + 0.1 \log 0.1]$$

$$= 1.156 \text{ bits/symbol.}$$

Entropy of the Receiver:

$$P(Y_1) = 0.28 + 0.04 + 0.04 = 0.36.$$

$$P(Y_2) = 0.28 + 0.08 + 0.02 = 0.38.$$

$$P(Y_3) = 0.14 + 0.08 + 0.04 = 0.26.$$

$$H(Y) = -[0.36 \log 0.36 + 0.38 \log 0.38 + 0.26 \log 0.26]$$

$$= 1.566 \text{ bits/symbol.}$$

Joint Entropy:-

$$H(X, Y) = -[0.28 \log 0.28 + 0.28 \log 0.28 + 0.14 \log 0.14]$$

$$+ 0.04 \log 0.04 + 0.08 \log 0.08 + 0.08 \log 0.08$$

$$+ 0.04 \log 0.04 + 0.02 \log 0.02 + 0.04 \log 0.04]$$

$$= 2.678 \text{ bits/symbol.}$$

Conditional Entropies:-

$$H(X|Y) = H(X, Y) - H(Y) = 1.112 \text{ bits.}$$

$$H(Y|X) = H(X, Y) - H(X) = 1.522 \text{ bits}$$

Mutual Information

$$I(X, Y) = H(X) + H(Y) - H(X, Y)$$

$$= 0.644 \text{ bits/message.}$$

Noise Bandwidth of A system

The Noise Bandwidth of a practical system is the Bandwidth of an ideal system which passes the same noise power as does the practical system.

Let $G_i(\omega)$, $H(\omega)$ and $G_o(\omega)$ be the P/p PSD, Transfer function and the o/p PSD of the system respectively.

$$\therefore G_o(\omega) = |H(\omega)|^2 \cdot G_i(\omega).$$

The mean square value of the output process of the system is

$$E[Y^2(H)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_o(\omega) d\omega.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 G_i(\omega) d\omega$$

Let the input be noise of constant PSD ' k ' W/Hz .

$$\therefore E[Y^2(H)] = \frac{k}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 G_i(\omega) d\omega.$$

Let the input be noise of constant PSD ' k ' W/Hz .

Now, define B_N as a Bandwidth with reference to a frequency $\omega = \omega_0$ as

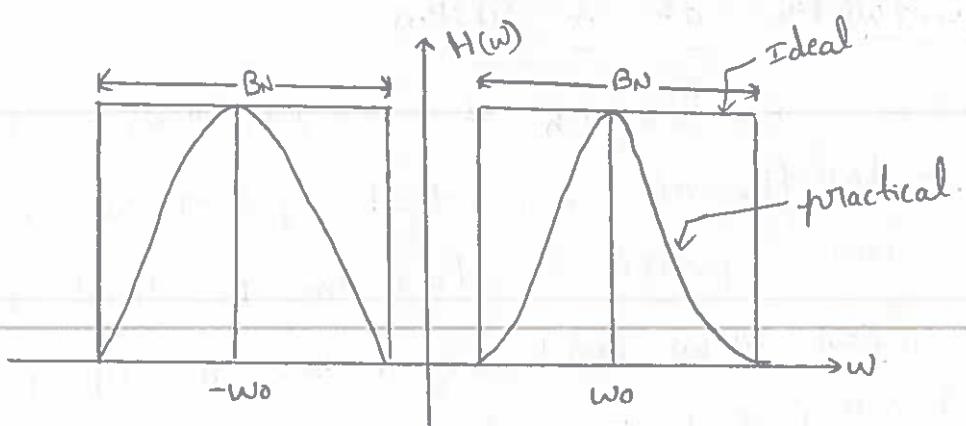
$$B_N = \frac{1}{2|H(\omega_0)|^2} \cdot \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega.$$

$$\therefore \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega = 2 \cdot B_N |H(\omega_0)|^2$$

$$\therefore E[Y^2(H)] = 2 \cdot \frac{k}{2\pi} B_N |H(\omega_0)|^2$$

Now, consider an ideal system of Bandwidth B_N , centered at $\omega = \omega_0$.

The Ideal and the practical systems considered are Unit-5, aim



Both systems functions are centered at same frequency
 $w = w_0$

Let the same above noise be applied as the input for ideal system also. The corresponding o/p PSD is $G_{no}(w) = |H(w_0)|^2 k$, because for ideal system, also. The corresponding o/p and it is practical.

The corresponding output power is

$$w_0 + \frac{B_N}{2}$$

$$\frac{1}{2\pi} \int_{w_0 - \frac{B_N}{2}}^{w_0 + \frac{B_N}{2}} |H(w)|^2 k dw = \frac{k}{2\pi} B_N |H(w_0)|^2.$$

The two sided output power is $2 \frac{k}{2\pi} B_N |H(w_0)|^2$. Thus, it is verified that the ideal system and practical system are allowing the same amount of noise power to pass through. Thus, the Bandwidth B_N of ideal system is the noise Bandwidth of the practical system.

$$\therefore \text{Noise Bandwidth } B_N = \frac{1}{2|H(w_0)|^2} \int_{-\infty}^{\infty} |H(w)|^2 dw.$$

Where $H(w_0)$ is $H(w)$ at its center frequency $w = w_0$. This is also given as

$$B_N = \frac{1}{|H(\omega_0)|^2} \int_0^\infty |H(\omega)|^2 d\omega.$$

Calculate the noise bandwidth of any RC LPF.
How it is related to its 3 dB Bandwidth:

The transfer function of an RCLPF is

$$H(\omega) = \frac{1}{1+j\omega RC}$$

$$\text{Noise Bandwidth } B_N = \frac{1}{2|H(0)|^2} \int_{-\infty}^{\infty} |H(f)|^2 df.$$

for RCLPF, $H_{max} = H(0) = 1$

$$\begin{aligned} \therefore B_N &= \frac{1}{2} \int_{-\infty}^{\infty} \left| \frac{1}{1+j2\pi f RC} \right|^2 df \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1+4\pi^2 f^2 R^2 C^2} df. \end{aligned}$$

Since the 3dB cut off frequency, $f_c = \frac{1}{2\pi RC}$.

$$\begin{aligned} B_N &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{1+(f/f_c)^2} df \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{f_c^2}{f_c^2 + f^2} df \\ &= \left[\frac{1}{2} f_c \tan^{-1}(f/f_c) \right]_{-\infty}^{\infty} = \frac{\pi f_c}{2}. \end{aligned}$$

since $f_c = \frac{1}{2\pi RC}$ $B_N = \frac{\pi}{2} \cdot \frac{1}{2\pi RC} = \frac{1}{4RC}$

The 3dB bandwidth f_1 is computed as follows,

$$[1+H(f)]_{f=f_1} = \frac{1}{\sqrt{2}} \quad \text{at } f=f_1 = 1/\sqrt{2}$$

$$\Rightarrow f_1 = f_c.$$

\therefore The Bandwidth of RC low pass filter

$$= \frac{1}{2\pi RC} = \frac{2}{\pi} \cdot \frac{1}{4RC}$$

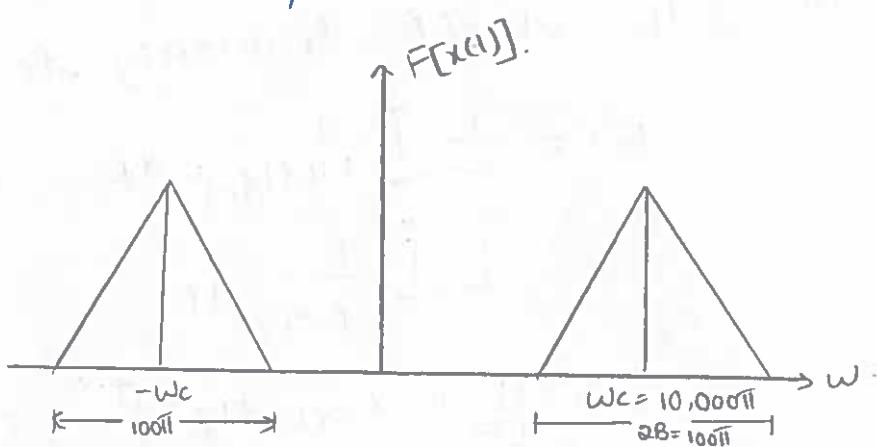
$$\Rightarrow (\text{BW})_{3\text{db}} = \frac{2}{\pi} \text{ Noise band width } B_N.$$

$$\Rightarrow B_N = \frac{\pi}{2} (\text{BW})_{3\text{db}}$$

Bandpass, Band Limited and Narrow Band processes:

A process is said to be a Band pass process if its Fourier transform is non-negligible only in a band of frequencies over an extent of $2B$, centered at an arbitrary frequency w_c .

If the band width $2B$ is small compared to the center frequency w_c , such a process is referred to as narrow band process.



The process is a Band limited process if its Fourier Transform is zero above a certain frequency w_c i.e. $x(w) = 0$ for $|w| > w_c$

In the mathematical analysis of communication systems, the channel noise $n(t)$ received along with the signal $s(t)$, is expressed as a Narrow Band process within the spectral range of the received signal.

The noise $n(t)$ is considered as narrow Band Gaussian process

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The Narrow Band representation of noise $n(t)$ is given as

$$n(t) = n_c(t) \cos \omega t - n_s(t) \sin \omega t$$

Here, $n_c(t)$ is referred to as phase component and $n_s(t)$ is referred to as quadrature component of noise and both are referred to as quadrature component of noise and.

$$\text{Let } n_c(t) = R(t) \cos \theta(t)$$

$$n_s(t) = R(t) \sin \theta(t)$$

$$\begin{aligned} n(t) &= R(t) [\cos \omega t \cos \theta(t) - \sin \omega t \sin \theta(t)] \\ &= R(t) \cdot \cos(\omega t + \theta(t)) \end{aligned}$$

where $R(t) = \sqrt{n_c^2(t) + n_s^2(t)}$ is referred to as envelope of noise and

$$\theta(t) = \tan^{-1} \left[\frac{n_s(t)}{n_c(t)} \right]$$

Properties of Quadrature Component of Noise:

1. $n_c(t)$ and $n_s(t)$ are jointly WSS, Gaussian processes with zero mean.
2. $E[n_c^2(t)] = E[n_s^2(t)]$
3. $n_c(t)$ and $n_s(t)$ are of same auto correlation function.
4. $E[n_c(t) \cdot n_s(t)] = 0$
5. $n_c(t)$ and $n_s(t)$ are as same PSD, given as $G_{nc}(f) = G_{ns}(f) = G_n(f_0 - f) + G_n(f_0 + f)$
6. In the representation $n(t) = n_c(t) \cos \omega t - n_s(t) \sin \omega t$, all the processes $n(t)$, $n_c(t)$ and $n_s(t)$ are narrow band processes.

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Describe the quadrature representation of narrow band noise.

Sol:

A narrowband band pass signal $g(t)$ is represented as

$$g(t) = g_I(t) \cos 2\pi f_c t + g_Q(t) \sin 2\pi f_c t$$

where $g_I(t)$ = in phase component of bandpass signal $g(t)$

$g_Q(t)$ = quadrature component of $g(t)$.

$$\eta_p(t) = n_I(t) + j \cdot n_Q(t)$$

$$\tilde{n}(t) = \eta_p(t) \exp(-j2\pi f_c t)$$

The complex envelop

$$\tilde{n}(t) = n_I(t) + j \cdot n_Q(t)$$

$$\therefore n_I(t) + j n_Q(t) = [n_I(t) + j n_Q(t)] \exp(-j2\pi f_c t).$$

$$n_I(t) = n_I(t) \cos 2\pi f_c t + n_Q(t) \sin 2\pi f_c t.$$

$$n_Q(t) = n_I(t) \cos 2\pi f_c t - n_I(t) \sin 2\pi f_c t.$$

Consider

$$n_I(t) \cdot \cos 2\pi f_c t = n_I(t) \cos^2 2\pi f_c t + n_I(t) \times \sin^2 2\pi f_c t$$

$$\text{Hence } n_Q(t) \cdot \sin 2\pi f_c t = n_I(t) \cos 2\pi f_c t \times \sin 2\pi f_c t - n_I(t) \sin^2 2\pi f_c t$$

$$\therefore n_I(t) = n_I(t) \cos 2\pi f_c t - n_Q(t) \sin 2\pi f_c t$$

This is called quadrature representation of narrow band noise.

The above $n_I(t)$ can be represented as

$$n(t) = r(t) [\cos 2\pi f_c t + q(t)]$$

$$r(t) = [n_I^2(t) + n_Q^2(t)]^{1/2}$$

which is envelop of noise and is Rayleigh distributed. $\phi(t)$ is the phase of $n(t)$ and is uniformly distributed.